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Shadowing Chaos Within Turbulence

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Final Technical Report

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Abstract

The first focus is on parametrization and decomposition of turbulence in geophysical flows. Such flows are described by 3D Euler/Navier-Stokes equations with Boussinesq stratification and uniform rotation. Within the context of rotation in thin atmospheric layers, we demonstrate that geophysical turbulence splits between 2D turbulence for the geostrophic component and phase turbulence for the ageostrophic component. By solving low-dimensional dynamical systems, we obtain exact formulas for phase turbulence and phase scrambling in atmospheric dynamics. Numerically, this leads to rigorous non-stiff operator splitting algorithms for geophysical flows. The second focus is on inertial stability of such algorithms. The new concept of inertial stability guarantees that long time simulation of turbulence via scientific computing is indeed statistically relevant to real physical turbulence.

1 Research Accomplishments

1.1 Global Regularity and Integrability of 3D Euler and Navier-Stokes Equations for Uniformly Rotating Fluids

There are many important engineering and geophysical problems in which rotation significantly modifies properties of fluid flows. In particular, large-scale atmospheric and oceanic flows are dominated by rotational effects and the impact of shallowness on the large scale dynamics. These problems attracted attentions of many physicists and mathematicians beginning with works of Laplace [1775-1779], Kelvin [1880], Poincaré [1910] (see Greenspan [1968] and Pedlosky [1987] for more references). Long-time computation of geophysical flows using unmodified Euler equations is prohibitive due to severe accuracy and time step restrictions. The disparity of time scales leads to problems in the numerical solution of the equations because the Courant number is determined by the fastest time scale $1/\Omega_0$ (Ω_0- angular velocity of the background rotation) and therefore limits the time step which makes explicit solution impractical (Browning, Holland, Kreiss and Worley [1990]). One of the major difficulties encountered is the influence of the oscillations generated by the rotation (see, for instance Bourgeois and Beale [1994], Lions, Temam and Wang [1994], Babin, Mahalov and Nicolaenko [1995a] and [1995b], Chemin [1995], Grenier [1995a] and [1995b] and more references in the cited papers, see also the list of references). We develop here the techniques which allow us to overcome this difficulty and in fact, the oscillations help to solve the problem explicitly.

We consider flows uniformly rotating with a constant angular velocity Ω_0 about the x_3 axis. Let L_0 be the characteristic horizontal length scale and U_0 be the characteristic velocity. The problem is made dimensionless by referring lengths to L_0, velocities to U_0 and time to L_0/U_0, which leads to the introduction of a dimensionless rotation rate Ω ($Ro = 1/\Omega$ is the Rossby number), defined by $\Omega = \frac{\Omega_0 L_0}{U_0}$.

In a frame of reference rotating with constant angular velocity Ω about the x_3 axis, the inviscid 3D Euler equations in the dimensionless variables have the form

$$\partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U} + 2\Omega \mathbf{J}\mathbf{U} \;=\; -\nabla p, \qquad (1)$$

$$\nabla \cdot \mathbf{U} \;=\; 0.$$

Here $\mathbf{U} = (U^1, U^2, U^3)$ is the velocity field, $\mathbf{J}$ is the rotation matrix, $2\Omega\mathbf{J}\mathbf{U}=2\Omega e_3 \times \mathbf{U}$ is the Coriolis force term and p is a modified pressure. We consider solutions of Eq. (1) $2\pi a_j$-periodic in every x_j taken along Cartesian axes e_j, $j = 1,2,3$; $a_1 = 1, 0 < a_2 \leq a_{21}, 0 < a_3 \leq a_{31}$. The linear version of (1) was extensively studied by Poincaré in 1910 with the introduction of Poincaré's velocity variables and Poincaré's inertial waves. Poincaré's waves are well-known in the geophysical literature (e.g. Greenspan [1968]).

In our previous work (Babin, Mahalov and Nicolaenko [1995a]) we have used Poincaré velocity variables and averaging Euler equations in time, derived new extended 2D/reduced 3D Euler/Navier-Stokes equations for the Poincaré's velocity variables. The structure of these equations is determined by the structure of resonant cones in the Fourier space of wavenumbers; after time averaging over the fast time scale $1/\Omega$, these resonant cones generate a non-trivial limit to the Euler equations as $\Omega \to +\infty$. The method is closely related to time-averaging methods in celestial mechanics (Poincaré [1892], Moser [1973], Siegel & Moser [1971]). In this work we resolve the small divisor problem for resonances in rotating 3D Euler equations. The resonant set in the Fourier space of wavenumbers is the intersection of five-dimensional cones with a lattice in $\mathbf{Z}^9$ which depends on a_2, a_3; we call this set resonant cones. Through a remarkable factorization of the corresponding small divisor, we prove that the resonant cones in Fourier space further reduce to the union of four three-dimensional cones if the spatial domain itself is non-resonant; that is the ratios a_2, a_3 of lengths of edges of the period parallelepiped are non-resonant (*corresponding resonant domain ratios* $1/a_2^2, 1/a_3^2$ *are of zero Lebesgue measure in* $\mathbf{R}$, *we assume everywhere below that such domain ratios are non-resonant*). Therefore aspect ratios are non-resonant with probability one. An important case of non-resonant domains is small a_3 (thin domains). We derive uniform estimates for the small divisor in the complement of resonant cones, for non-resonant domains. These are essential for the sharp estimates of the error (in terms of the anisotropic Rossby number) in our strong convergence theorems. Without such small divisor estimates, one can only state non-uniform convergence theorems with an error estimate $o(1)$, such as done in Grenier [1995a] and [1995b]. The resolution of the small divisor

problem is at the very heart of our global existence theorems for 3D rotating Euler and Navier-Stokes equations presented in this work.

In this work we study small anisotropic Rossby number flows. The anisotropic Rossby number Ro_a is defined as:

$$Ro_a = a_3/\Omega = a_3 Ro. \tag{2}$$

The vertical aspect ratio a_3 in general is not assumed to be small; as a matter of fact, its smallness improves error estimates in the convergence theorems. The anisotropic number is small in the following four important situations:

(a) rapidly rotating flows ($\Omega >> 1$, but $a_3 = O(1)$).

(b) rotating flows in shallow layers ($a_3 << 1$ but $\Omega = O(1)$).

(c) intermediate: $Ro_a = a_3/\Omega = a_3 Ro << 1$.

(d) combined: $a_3 << 1$, $1/\Omega << 1$.

The small anisotropic Rossby number limit ($Ro_a << 1$) covers uniformly *all* four cases (a)-(d). The limit $Ro_a << 1$ is not a mathematical ethereal limit never reached in "real physical" geophysical flows; on the contrary, our rigorous error estimates are of order Ro_a where the anisotropic Rossby number $Ro_a = a_3/\Omega = a_3 Ro$ is of order $10^{-3} - 10^{-4}$ for important classes of atmospheric and oceanic flows; for example, $a_3 \approx 10^{-2}$ and $Ro \approx 0.1$ for large scale atmospheric flows (synoptic scales) at midlatitudes which implies $Ro_a \approx 10^{-3}$ (e.g. Holton [1992]).

For small anisotropic Rossby number flows, we rigorously show that solutions of 3D Euler/Navier-Stokes equations can be decomposed as

$$\mathbf{U}(t, x_1, x_2, x_3) = \tilde{\mathbf{U}}(t, x_1, x_2) + \mathbf{V}(t, x_1, x_2, x_3) + \mathbf{r}$$

where $\tilde{\mathbf{U}}$ is a solution of the 2D Euler system with vertically averaged initial data; $\mathbf{r}$ is a remainder, its Sobolev norm is estimated from above by a quantity proportional to Ro_a, for smooth enough initial data.

The norm of $\mathbf{V}(t)$ is conserved, $||\mathbf{V}(t)|| = ||\mathbf{V}(0)|| = ||\mathbf{U}(0) - \tilde{\mathbf{U}}(0)||$, therefore *it is not small.* The vector field $\mathbf{V}(t)$ is related to $\mathbf{v}(t)$ by a Van der Pol type transformation based on the Poincaré propagator and $\mathbf{v}(t)$ is a solution of a new system which we call Extended

Euler/Navier-Stokes system. The Poincaré propagator (Poincaré 1910) is the unitary group solution $\mathbf{E}(\Omega t)$ ($\mathbf{E}(0) = \mathbf{Id}$ is the identity) to the Poincaré problem:

$$\partial_t \Phi + 2\Omega \mathbf{J}\Phi = -\nabla p,\ \nabla \cdot \Phi = 0 \tag{3}$$
$$\Phi(t) = \mathbf{E}(-\Omega t)\Phi(0).$$

We use Fourier series expansions for velocity fields $\mathbf{U}(x) = (U^1(x), U^2(x), U^3(x))$

$$\mathbf{U}(x) = \sum_n \exp(i(n_1 x_1 + n_2 x_2/a_2 + n_3 x_3/a_3))\mathbf{U}_n \tag{4}$$

where $\mathbf{U}_n$ are the Fourier coefficients, $(n_1, n_2, n_3) \in \mathbf{Z}^3$ are wavenumbers. We assume that functions have zero average over the periodic parallepiped.

Poincaré [1910] solutions can be written in Fourier space as:

$$\mathbf{E}_n(\Omega t) = \{\mathbf{E}(\Omega t)\}_n = \cos(\frac{n_3}{a_3|\check{n}|}2\Omega t)\mathbf{I} + \frac{1}{|\check{n}|}\sin(\frac{n_3}{a_3|\check{n}|}2\Omega t)\mathbf{R}_n \tag{5}$$

where the matrix $i\mathbf{R}_n$ is the Fourier transform of the *curl* operator; we denote $|\check{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$. The mathematical theory of the Poincaré problem (3) has attracted a considerable amount of attention starting from the work of Sobolev [1954].

The Poincaré velocity $\mathbf{v}(t)$ is related to $\mathbf{V}(t)$ by $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$, which is a canonical unitary transformation preserving energy, helicity and divergence free property. For fields which depend only on x_1, x_2 but not on x_3, $\mathbf{E}(\Omega t) = \mathbf{Id}$. Before any averaging the equations for $\mathbf{v}$ are Euler-like, but with time-dependent coefficients. The extended Euler equations (which are obtained after time-averaging for non-resonant domains) are for the velocity fields which are three-dimensional and three-component (3D-3C: three components, dependence on three variables x_1, x_2, x_3). *The extended system possesses infinitely many conservation laws.* These conserved quantities are adiabatic invariants for the classical 3D Euler equations in a rotating frame in the small anisotropic Rossby number case. If the vertically averaged initial data are reflectionally symmetric (that is satisfy zero flux boundary conditions in the half-sized box), we can fully integrate the extended system and obtain the exact solutions for $\mathbf{v}(t)$ and $\mathbf{V}(t)$. We underline that the splitting is global, the initial data are not assumed to be small and strong vertical shearing is allowed.

As usual we denote x_3-averaging of a function U by $\overline{U}$, and for vector fields $\overline{\mathbf{U}}(x_1, x_2) = \frac{1}{2\pi a_3}\int_0^{2\pi a_3} \mathbf{U}(x_1, x_2, x_3)dx_3$. We prove the theorem on behavior of solutions of 3D Euler equations which gives a rigorous sense to the Taylor-Proudman theorem for time-dependent flows. Namely, we have

Theorem 1.1.

Let a_2 be non-resonant. Let $\mathbf{U}(t)$ be an exact solution of 3D Euler equations (1) with initial data $\mathbf{U}(0) = \mathbf{U}(0, x_1, x_2, x_3) = \mathbf{U}_0(x_1, x_2, x_3)$. Let $\sigma - 9 \geq \alpha > 3/2$, let $M_{0\sigma} > 0$. Then there exists $T_1 > 0$ such that if $\|\mathbf{U}(0)\|_\sigma \leq M_{0\sigma}$, then

$$\|\overline{\mathbf{U}}(t) - \tilde{\mathbf{U}}(t)\|_\alpha^2 \leq CRo_a \tag{6}$$

for $0 \leq t \leq T_1$; C, T_1 depend only on $M_{0\sigma}$. Here $\|\cdot\|_\sigma$ is the norm in a Sobolev space H_σ (of divergence free periodic functions) and $\tilde{\mathbf{U}}(t)$ is a solution of 2D-3C Euler equations with vertically averaged initial data $\tilde{\mathbf{U}}(0) = \tilde{\mathbf{U}}(0, x_1, x_2) = \tilde{\mathbf{U}}_0(x_1, x_2)$; the 2D-3C Euler equations are for velocity fields independent of x_3:

$$\partial_t \tilde{\mathbf{U}} + \tilde{\mathbf{U}} \cdot \nabla\tilde{\mathbf{U}} = -\nabla\tilde{p},\ \nabla \cdot \tilde{\mathbf{U}} = 0 \tag{7}$$

where $\tilde{\mathbf{U}} \cdot \nabla$ is the classical 2D Euler advection operator. In particular,

$$\partial_t \tilde{U}^3 + (\tilde{U}^1 \frac{\partial}{\partial x_1} + \tilde{U}^2 \frac{\partial}{\partial x_2})\ \tilde{U}^3 = 0. \tag{8}$$

This theorem shows that the exact x_3-averaged vector field $\bar{\mathbf{U}}(t)$ is close to a solution $\tilde{\mathbf{U}}(t)$ of 2D-3C Euler equations (7)-(8) with vertically averaged initial data $\tilde{\mathbf{U}}(0) = \bar{\mathbf{U}}_0(x_1, x_2)$. We use the terminology of W.C. Reynolds and Kassinos [1994]. In their terminology 2D-3C fields have three components and depend on two variables x_1 and x_2; 3D-3C fields have three components and depend on three variables x_1, x_2 and x_3. A direct corollary of this theorem is separate approximate conservation of the energy of components $\bar{\mathbf{U}}(t)$ and $\mathbf{U}(t) - \bar{\mathbf{U}}(t)$ of the flow. This implies, in particular, that $\mathbf{U}(t) - \bar{\mathbf{U}}(t)$ is not small. The existence of classical conserved quantities for 2D Euler equations (integrals of functions of the vertical component of *curl*) and the above theorem imply existence of approximate

conservation laws (adiabatic invariants, Arnold 1978) for 3D Euler equations in the case of small Ro_a.

The difference $\mathbf{U}(t) - \tilde{\mathbf{U}}(t)$ (or $\mathbf{U}(t) - \overline{\mathbf{U}}(t)$) can be rigorously approximated by $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$ where $\mathbf{v}(t)$ is a solution of the extended Euler equations. These equations are for the velocity fields which are three-dimensional and three-component (3D–3C). They have the following form, where $\tilde{\mathbf{U}}$ is the solution of the 2D-3C Euler equations (7)-(8) with x_3- averaged initial data:

$$\partial_t \mathbf{v} = \mathbf{B}_{ex}(\tilde{\mathbf{U}}, \mathbf{v}); \tag{9}$$

here $\mathbf{B}_{ex}(\tilde{\mathbf{U}}, \mathbf{v})$ is a bilinear operator and $\overline{\mathbf{v}} = 0$. After $\tilde{\mathbf{U}}$ is found from Eqs. (7)-(8), $\mathbf{v}$ is obtained from Eq. (9) which is a linear equation for $\mathbf{v}$ with time and space dependent variable coefficients determined by $\tilde{\mathbf{U}}$.

The bilinear operator $\mathbf{B}_{ex}(\tilde{\mathbf{U}}, \mathbf{v})$ is a non-local bilinear operator for the x_3-dependent $\mathbf{v}$ field and $\tilde{\mathbf{U}}$. It comes from a non-local restriction of Euler-like operators to a special class of wavenumber interactions determined by the resonance conditions. These Euler-like operators were explicitly written and its properties discussed in Babin, Mahalov and Nicolaenko [1995a]. Similar extended equations were written for the Navier-Stokes equations.

The remarkable property of the extended system (9) is that (for generic, non-resonant a_2, a_3) *it splits into an infinite sequence of independent (uncoupled) subsystems of linear 8-component ordinary differential equations (ODE's)* for the corresponding (divergence-free) Fourier modes. For a given index (n_1, n_2, n_3), $n_3 \neq 0$ they describe interactions of the quadruplet $\mathbf{U}_{(\pm n_1, \pm n_2, n_3)}$ of the Fourier modes. Each subsystem couples 8 non-autonomous ODE's. The equations are (in the case of non-resonant domains)

$$\partial_t \mathbf{v}_n = i \sum_{\substack{m_1 = \pm n_1,\, m_2 = \pm n_2 \\ m_3 = n_3,\, k + m = n}} [-\mathbf{P}_n(\tilde{\mathbf{U}}_k \cdot \check{m})\mathbf{v}_m + \frac{\check{n}}{2|\check{n}|^2} \times ((\check{k} \times \tilde{\mathbf{U}}_k) \times \ (\check{m} \times \mathbf{v}_m))], \tag{10}$$

where $\check{n} = (n_1, n_2/a_2, n_3/a_3)$ and similarly for $\check{k}$, $\check{m}$; $\mathbf{P}_n$ is the matrix of Leray projection onto the space of divergence free periodic vector fields. The coefficients of this system are determined by the solution $\tilde{\mathbf{U}}(t, x_1, x_2)$ of 2D-3C Euler equations (7)-(8). The equations

are very simple and can be used for numerical computations of 3D rotating flows, since the subsystems are decoupled in a very convenient way for parallel computing.

Every subsystem (10) preserves energy and helicity separately, which gives infinitely many conservation laws for the whole system; moreover, the 3D H_s-Sobolev norms, for every s, are conserved for $\mathbf{v}$ (including enstrophy). The existence of infinite number of conservation laws for the extended Euler equations is in contrast to the classical 3D Euler equations, where only energy and helicity are conserved (Serre [1984a] and Serre [1984b]). *These conserved quantities of the extended Euler equations are the approximate adiabatic invariants (Arnold 1978) of 3D Euler equations in the small anisotropic Rossby number situation.*

Being so elegant and even explicitly integrable as we describe below, the extended equations describe exact 3D Euler flows with high accuracy if Ro_a is small. We show that $\mathbf{U}(t)$ can be rigorously approximated by $\tilde{\mathbf{U}}(t) + \mathbf{V}(t)$ where $\mathbf{v}(t)$ is a solution of the extended equations (9), (10). The vector field $\mathbf{v}(t)$ is related to $\mathbf{V}(t)$ by means of the explicitly given linear unitary operator $\mathbf{E}(\Omega t)$, Eq. (3) (the Poincaré propagator), $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$.

We prove the following theorem delineating the structure of exact solutions of 3D Euler equations in the small anisotropic Rossby number situation; effectively, we show that $\tilde{\mathbf{U}} + \mathbf{V} \to \mathbf{U}$ as $Ro_a \to 0$, uniformly with respect to initial data and strongly in Sobolev norms, and we give a power estimate of the remainder:

Theorem 1.2.

Let a_2, a_3 be non-resonant, let $\sigma - \alpha \geq 35$, $\alpha > 3/2$, $M_{0\sigma} > 0$. Let $\|\mathbf{U}(0)\|_\sigma \leq M_{0\sigma}$. Let $\mathbf{U}(t)$ be an exact solution of 3D Euler equations (1) with $\mathbf{U}(0) = \mathbf{U}_0(x_1, x_2, x_3)$. Let $\mathbf{E}(\Omega t)$ be the Poincaré propagator, $\mathbf{v}(t)$ be the solution of the extended Euler system with initial data $\mathbf{v}(0) = \mathbf{U}(0) - \tilde{\mathbf{U}}(0)$. Then the difference $\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)$ satisfies the estimate

$$||\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)||_\alpha^2 \leq C Ro_a \text{ for } 0 \leq t \leq T_1 \tag{11}$$

where T_1, C depend on $M_{0\sigma}$.

Similar estimate holds for solutions of 3D Navier-Stokes and extended Navier-Stokes

equations (uniformly in the viscosity ν, $0 \leq \nu \leq 1$).

Using Theorem 1.2 on approximation of solutions of 3D rotating Euler equations by solutions of extended Euler equations we have proven regularity of solutions of 3D Euler equations with arbitrary large initial data on arbitrary long time intervals in the small anisotropic Rossby number situation:

Theorem 1.3.

Let the domain be nonresonant; $M > 0$, $T^* > 0$ be arbitrary large. Then there exists $Ro_a^* = Ro_a^*(M, T^*)$ such that if $||\mathbf{U}(0)||_{38} \leq M$ and $0 < a_3/\Omega \leq Ro_a^*$ there exists a unique regular solution $\mathbf{U}(t), 0 < t < T^*$ of 3D Euler equations which belongs to H_{38} as $0 \leq t \leq T^*$. For M fixed, $T^* \to +\infty$ with $Ro_a^* \to 0$. Simultaneously we can take arbitrary large (but bounded) sets of initial data: $M \to +\infty$ if $Ro_a^* \to 0$.

The problem of global regularity of solutions of 3D Navier-Stokes equations has been extensively studied by many mathematicians and still is an outstanding unsolved problem of the applied analysis. Using the detailed description of dynamics of the extended Navier-Stokes equations and error estimates (analogue of Theorem 1.2 for the Navier-Stokes equations) we have solved the problem in the situation of a small anisotropic Rossby number . Namely, we consider the 3D Navier-Stokes equations with the time-independent forcing $\mathbf{F}$ on the infinite time interval :

$$\begin{aligned} \partial_t \mathbf{U} + \nu curl curl \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + 2\Omega \mathbf{J} \mathbf{U} &= -\nabla p + \mathbf{F}, \qquad (12) \\ \nabla \cdot \mathbf{U} &= 0. \end{aligned}$$

We prove

Theorem 1.4.

Let the domain be nonresonant. Let $\mathbf{F} \in H_{38}$, $M > 0$. There exists $Ro_a^0(M, \nu, ||\mathbf{F}||_{38})$ such that if $||\mathbf{U}(0)||_1 \leq M$ and $0 < a_3/\Omega \leq Ro_a^0$ then there exists a unique regular solution $\mathbf{U}(t)$, $0 \leq t < +\infty$ of 3D Navier-Stokes equations which belongs to H_1 as $t \geq 0$, $||\mathbf{U}(t)||_1 \leq C_1(M, \nu, ||\mathbf{F}||_{38})$, and to H_{38} as $t > 0$; and $||\mathbf{U}(t)||_{38} \leq C_{38}(M, \nu, ||\mathbf{F}||_{38}, t_0)$ for

every $t \geq t_0 > 0$. (The constants C_1, C_{38} are uniform in time t.)

In Theorem 1.4 we can take arbitrary large M, $||\mathbf{F}||_{38}$ and a solution is regular if $Ro_a = a_3/\Omega$ is small enough; but the question whether or not for a small *fixed* Ro_a solutions with arbitrary large initial data blow up in H_1 in finite time is still open. In all cases, the energy estimate $\frac{1}{T}\int_0^T ||\mathbf{U}(t)||_1^2 dt \leq \nu^{-1}||\mathbf{U}(0)||^2/T + \nu^{-2}||\mathbf{F}||_{-1}^2$ holds.

If we take $T \geq T^* = ||\mathbf{U}(0)||_0^2/\nu$, $M_1^2 = 2\nu^{-2}||\mathbf{F}||_{-1}^2 + 2$ then $||\mathbf{U}(t_c)||_1 \leq M_1$ for some $t_c \in [0, T)$. Then applying Theorem 1.4 with a shift in time and using the equality of a weak solution of 3D Navier-Stokes equations with the regular one we conclude that *a weak solution with arbitrary large initial data in H_0 is regular for $t > T^*$ and is attracted to the global attractor if $0 < Ro_a \leq Ro_a^0(M_1, \nu, ||\mathbf{F}||_{38})$.* The regularization time T^* depends only on the L_2-norm of initial data and the critical Rossby number Ro_a^0 does not depend on initial data at all.

We do not need to impose conditions of the thin domain type and do not assume that initial data are close to 2D initial data in some Sobolev space. Furthermore, the dependence of initial velocity fields on x_3 can be arbitrary and, in particular, we allow strongly sheared flows in x_3 (provided the fields are smooth). *The parameter a_3 need not be small; flows corresponding to initial conditions may have strong x_3-dependence (strong x_3- shearing) and large U^3-component.*

Remarkably, if the vertically averaged initial data $\tilde{\mathbf{U}}(0)$ are reflectionally symmetric, we can fully integrate the extended Euler system. Now we give explicit formulas which express solutions of the extended Euler equations (9), (10) in terms of its time-dependent coefficients. The corresponding $\tilde{\mathbf{U}}(t, x_1, x_2) + \mathbf{V}(t, x_1, x_2, x_3)$ are exact explicit solutions of 3D Euler equations in the small anisotropic Rossby number limit.

Consider the case when x_3-averaged initial data have the reflection symmetry $\bar{U}^j(-x_i) = \bar{U}^j(x_i), i \neq j$, $\bar{U}^i(-x_i) = -\bar{U}^i(x_i), i, j = 1, 2$ at $t = 0$; it corresponds to the usual Euler zero-flux boundary condition $\bar{U}^j = 0$ as $x_j = 0$ and $x_j = \pi a_j, j = 1, 2$ (which are the natural boundary conditions for the 2D Euler equations in a rectangular domain) at the boundary of the half-sized box. Moreover, we take $\overline{U}^3(0)$ odd with respect to the reflection

symmetries in x_1 and x_2; this corresponds to the no-slip condition for the third averaged component. For (n_1, n_2, n_3) we denote $L(n_1, n_2, n_3) = (-n_1, -n_2, n_3)$, $n' = (n_1, n_2, 0)$.

In the case of symmetrical averaged data the extended Euler equations decouple into systems of ODE's describing interactions between the Fourier coefficients $\mathbf{v}_n$ and $\mathbf{v}_{Ln}$. Let

$$\xi_n(t) = \frac{1}{2}(curl\tilde{\mathbf{U}})_{2n'} \cdot e_3$$

be the Fourier coefficient of the vertical component of *curl* corresponding to the wavenumber $2n' = (2n_1, 2n_2, 0)$ and

$$\eta_n(t) = \frac{n_1^2 + n_2^2/a_2^2 - n_3^2/a_3^2}{|\check{n}|}\tilde{U}^3_{2n'}(t)$$

where $\tilde{U}^3_{2n'}(t)$ is the third component of the solution of 2D-3C Euler system, $|\check{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$. The functions ξ and η are real-valued.

Then Eqs. (10) split into equations of the form

$$\partial_t \begin{pmatrix} \mathbf{v}_n \\ \hat{\mathbf{v}}_n \end{pmatrix} = \frac{n_3}{a_3|\check{n}|^2}\left(\xi_n(t)\mathbf{R}_n' - \eta_n(t)\; i|\check{n}|\mathbf{I}'\right)\begin{pmatrix} \mathbf{v}_n \\ \hat{\mathbf{v}}_n \end{pmatrix} \tag{13}$$

where $\hat{\mathbf{v}}_n = L\mathbf{v}_{Ln} = (-v^1_{Ln}, -v^2_{Ln}, v^3_{Ln})$ and

$$\mathbf{I}' = \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix},\; \mathbf{R}_n' = \begin{pmatrix} 0 & \mathbf{R}_n \\ \mathbf{R}_n & 0 \end{pmatrix},\; \mathbf{R}_n = \begin{pmatrix} 0 & -n_3/a_3 & n_2/a_2 \\ n_3/a_3 & 0 & -n_1 \\ -n_2/a_2 & n_1 & 0 \end{pmatrix}. \tag{14}$$

Here $\mathbf{I}$ is the identity matrix and $i\mathbf{R}_n$ is the *curl* in Fourier space; $i\mathbf{I}'$ and $\mathbf{R}_n'$ are skew-symmetric matrices which *commute.* The commutativity property allows us to write general solutions of the system (13) explicitly. The solution is given by

$$\begin{pmatrix} \mathbf{v}_n(t) \\ \hat{\mathbf{v}}_n(t) \end{pmatrix} = \exp\left(\frac{n_3}{a_3|\check{n}|^2}[\tau_1(t)\mathbf{R}_n' - \tau_2(t)\; i|\check{n}|\mathbf{I}']\right)\begin{pmatrix} \mathbf{v}_n(0) \\ \hat{\mathbf{v}}_n(0) \end{pmatrix} \tag{15}$$

where

$$\tau_1(t) = \int_0^t \xi_n(s)ds,\; \tau_2(t) = \int_0^t \eta_n(s)ds. \tag{16}$$

We note that

$$\exp\left(\frac{n_3}{a_3|\check{n}|^2}\tau_1(t)\mathbf{R}'_n\right) = \cos\left(\frac{n_3}{a_3|\check{n}|}\tau_1(t)\right)\mathbf{Id} + \sin\left(\frac{n_3}{a_3|\check{n}|}\tau_1(t)\right)\frac{1}{|\check{n}|}\mathbf{R}'_n,$$

$$\exp\left(\frac{n_3}{a_3|\check{n}|^2}\tau_2(t)i\mathbf{I}'\right) = \cos\left(\frac{n_3}{a_3|\check{n}|}\tau_2(t)\right)\mathbf{Id} + \sin\left(\frac{n_3}{a_3|\check{n}|}\tau_2(t)\right)i\mathbf{I}' \tag{17}$$

where $\mathbf{Id} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$, $|\check{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$; we always take divergence-free $\mathbf{v}_n$. For Navier-Stokes equations without force, the right hand side of (15) has to be multiplied by $\exp(-\nu|\check{n}|^2 t)$.

We recall that $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$. An important observation is that the Poincaré propagator $\mathbf{E}(\Omega t)$ commutes with $\exp(\frac{n_3}{a_3|\check{n}|^2}\tau_1(t)\mathbf{R}'_n)$ and $\exp(\frac{n_3}{a_3|\check{n}|^2}\tau_2(t)i|\check{n}|\mathbf{I}')$ and, therefore, the phases $2\Omega t$, $\tau_1(t)$ and $\tau_2(t)$ are added in the final formulas for $\mathbf{V}(t)$. Thus, the vector field $\mathbf{V}(t)$ which approximate $\mathbf{U}(t) - \bar{\mathbf{U}}(t)$ is phase-locked to the phases $2\Omega t$, $\tau_1(t)$ and $\tau_2(t)$. The phases $\tau_1(t)$ and $\tau_2(t)$ are associated with vertically averaged vertical vorticity $curl\bar{\mathbf{U}}(t) \cdot e_3$ and velocity $\overline{U}^3(t)$; the latter is multiplied (in Fourier space) by the wave operator $n_1^2 + n_2^2/a_2^2 - n_3^2/a_3^2$ and $1/|\check{n}|$ (smoothing). The phase formulas (17) are strikingly similar to the Poincaré propagator formula (5); except that they describe mode coupling and $2\Omega t$ is replaced by $\tau_1(t)$, respectively $\tau_2(t)$, which are phases associated to the passive scalars $curl\bar{\mathbf{U}}(t) \cdot e_3$, respectively $\bar{U}^3(t)$. The latter are passively advected scalars by 2D turbulence.

Our exact solutions completely solve the outstanding problem of parametrization of 3-D rotation dominated turbulence in terms of pure 2-D turbulence of vertically averaged fields, in the limit of small anisotropic Rossby number Ro_a. Three-dimensional rotating turbulence in the small anisotropic Rossby number situation decouples into irreducible phase-locked turbulence for $\mathbf{V}(t, x_1, x_2, x_3)$ and 2D turbulence for vertically averaged fields. Statistics of 2D-2C turbulence are the sole player for $\bar{\mathbf{U}}(t)$. This vindicates all current research on intermittencies, vortex dynamics and inverse cascades in 2-D turbulence. It makes very relevant all the investigations of 2D vortex dynamics and statistics of 2D turbulence by Kraichnan, Montgomery, Saffman, Someria, Zabusky, McWilliams among many others.

Experimental studies, direct numerical simulations (DNS), large-eddy simulations (LES), and closure approximations have established that solid body rotation suppresses the nonlinear energy cascade from large scales to small scales. The effect of rotation is through "phase scrambling" for the wave phase. For MHD turbulence, Kraichnan [1965] pointed out that the propagation of the Alfvénic fluctuations disrupts phase relation and thereby may be expected, on the average, to decrease energy transfer. Similarly, uniform rotation causes plane waves to propagate with phase speed $2\Omega k_3/|k|$ resulting in "phase scrambling" thanks to strong dispersion and, on the average, decreases energy transfer. Our results on regularity of 3D Euler and Navier-Stokes equations reveal the regularizing effect of rotation; the mechanism of the regularization is the weakening of nonlinear interactions between almost all (non-resonant) Fourier modes through time-averaging. The mechanism principally differs from previously known regularizing mechanisms where viscosity plays the crucial role in regularization. The theorems proved in our work give a rigorous mathematical sense to nonlinear "phase scrambling".

1.2 Inertial Stability of Numerical Algorithms for Fluid Dynamics

Exponential attractors give a better answer for the dynamical and numerical significance of long-time flow simulation by supercomputers. Contrary to appearance, neither classical attractors, nor approximate inertial manifolds are uniformly continuous under perturbations; they are only upper semi-continuous. As a consequence, the classical attractor of a discretized numerical algorithm or nonlinear Galerkin scheme cannot uniformly converge to the exact (continuum) attractor (in the Navier-Stokes context). In contrast, exponential attractors are robust under perturbations (uniformly continuous) as proven for Galerkin approximations of 2D Navier-Stokes equation in Eden, Foias, Nicolaenko and Temam (1994). The existence and uniform convergence of approximate exponential attractors for semi-implicit and explicit discretization schemes for the Navier-Stokes equations are of particular interest.

Exponential attractors are enlargements of global attractors; they are in general strongly continuous with respect to perturbations, whereas the latter are usually only upper-semicontinuous.

An inertially stable algorithm is one which (i) admits a discretized exponential attractor and (ii) the latter uniformly converges to an exact exponential attractor for infinite resolution. An inertially stable algorithm shall preserve the statistics of turbulence. Inertial stability of an algorithm for classical 2D Navier-Stokes equations implies the same for small Rossby number rotating 3D Navier-Stokes equations. Algorithms implicit for the Stokes operator, yet fully explicit for the Euler advection term are in fact globally dissipative; even though energy and enstrophy are not preserved. Their algorithmic inertial stability condition is superior to that of algorithms semi-implicit in the advection term; even though the latter do preserve energy and are trivially globally dissipative. This is due to better control of turbulent fluctuating direct cascades of enstrophy and inverse cascades of energy by the first class of algorithms. Optimal inertial stability yields effective control of unresolved subgrid scales, and optimal resolution of large scale vortices in 3D small Rossby number rotating turbulence.

We have investigated "inertial stability" of numerical algorithms as the proper tool to ensure the uniform approximation of dynamical turbulent attractors. Our substantial results (Babin and Nicolaenko [1994], Babin, Mahalov and Nicolaenko [1996b]) within the context of two-dimensional Navier-Stokes turbulence shall provide a handsome pay-off in the context of geophysical turbulence, through the exact 2-D, 2-C "geostrophic" operators. We shall expand this effort.

Let us contrast "inertial algorithmic" stability versus classical stability concepts for numerical schemes.

For numerical accuracy, one traditionally considers the error estimates for *trajectories*. Exponential growth in time is generally present in the classical results. As shown in the most recent and sharpest estimates for the errors induced by both classical and nonlinear Galerkin methods on N.S. flows, (Devulder-Marion-Titi [1994], Rautmann [1980]), a non-linear Galerkin trajectory will shadow an exact one for N.S. flows up to a finite time, for a given tolerance. Of course, the sharper the algorithm, the longer the shadowing time. Such a separation is to be expected: even when there are *no* computational errors at all, generally a small perturbation of the initial condition can cause exponential splitting of two

trajectories.

Much work has been done on errors in estimating the global dynamical attractor A, for an extensive variety of approximation schemes; they are all based on the upper semi-continuity of the global attractor.

Exponential attractors give a better answer for the dynamical and numerical significance of long-time flow simulation by supercomputers. They are *continuous* with respect to the full Hausdorff (Federer, [1969]) set-distance, for "good" approximation schemes.
The existence of exponential attractors for turbulent Navier-Stokes flows has a major impact on their large scale computing. For instance, with spectral methods it is common that half of the modes (the small ones) carry less than 10^{-4} to 10^{-6} of the total energy. Hence the computer tends to be saturated by small wavelengths carrying little energy. Similarly, with finite differences, the unknown varies little between two successive grid points when $Dx \simeq 10^{-3}$. *Exponential attractors filter, in large-scale computing, these subgrid scale effects built into the discrete system.* A numerical algorithm must not only be vested with conventional stability; as a discretized large finite-dimensional dynamical system, *it must have built-in uniform stability of its own discretized exponential attractor.* We implement such a program based on the entirely new

Definition. *An approximate exponential attractor is the exponential attractor of an "inertially stable" scheme.*

Definition. *A scheme is "inertially stable" if and only if:*

(i) *it is stable, dissipative and possesses an exponential attractor M_N of its own. Here N is the dimension of the phase space of the scheme.*

(ii) *for all ϵ, there exists $N(\epsilon)$ so that:*

$$d_H(M_N, M) \leq \epsilon$$

(*Hausdorff distance*).

iii) *the constant of exponential convergence rate to M_N (for dynamics of the approximate scheme) is of the same order as estimates of that constant for the exact theoretical M.*

In ii), ϵ is the tolerance. iii) ensures that approximate dynamics lock onto the approximate M_N *as fast* as the exact dynamics onto M.

Corollary. *After time transients of the* same order, *the approximate and exact dynamics take place on* M_N *and* M.

Remark. There is no need to further compute M_N. After *uniform transients*, the dynamics of the scheme are on M_N. This is *not* true for convergence to A and/or A_N.

Eden, Foias, Nicolaenko and Temam [1994] have proven that the classical Galerkin scheme is "inertially stable" (this is a semi-discrete scheme where the time variable is kept continuous). *This is the litmus test for any other scheme*:

- **Determine which approximation schemes are vested with an approximate exponential attractor and are indeed inertially stable?**

The task is much less formidable than it appears, since the difficult questions of finite-time stability and dissipativity of whole classes of algorithms have already been resolved in the literature quoted above.

Our approach is a novel one for the effective control of unresolved subgrid scales in terms of long time integration of turbulence. As a payoff, statistics of turbulence are *uniformly preserved for long-time integration.* Also, at any time (especially long times in the DNS), there is a finite piece of an exact physical trajectory tracked by the computed one. Among concrete and more practical new results (Babin and Nicolaenko [1994], Babin, Mahalov, Nicolaenko [1996b]) we find that:

- inertial stability of a given algorithm crucially depends on the built-in *spatio-temporal physical scales* (including unresolved subgrid scales) of turbulence for the specific physical model computed (usually not true for classical algorithmic stability). For classical (incompressible) Navier-Stokes, one finds that the physical scales of the inertial range of turbulence, as well as the extent of that range, play a key role. Numerical turbulent attractor stability must be dealt with on a basis of specific classes of models: the physical scales of a turbulent boundary layer around an aircraft wing span differ from those of quasi 2-dimensional (2-D) geophysical flows.

One also finds that (at least for incompressible Navier-Stokes):

- the better the numerical scheme approximates *all conservation laws* of the underlying inviscid Euler equation, the more performing it is from the turbulent attractor stability

perspective. One conjectures this to be true for more general algorithmic models and plan to test it for geophysical models. For instance, semi-implicit (in the nonlinear advective term) schemes for Navier-Stokes do conserve energy exactly, but fail to preserve "enstrophy" (that is, the total energy of the vorticity field). They turn out to be much *worse* off than fully explicit (in the nonlinear advection term) schemes which *approximately yet uniformly preserve both energy and enstrophy.*

2 Publications Supported by the Grant and Bibliography

A. Babin and B. Nicolaenko (1994), *Inertially stable algorithms for Navier-Stokes equations*, Progress in Partial Differential Equations 3, Pitman Research Notes in Math., Vol. 314, p. 185-198.

A. Babin, A. Mahalov and B. Nicolaenko (1995a), *Long-Time Averaged Euler and Navier-Stokes Equations for Rotating Fluids*, In "Structure and Dynamics of Nonlinear Waves in Fluids", K. Kirchgässner and A. Mielke (eds), World Scientific, p. 145–157.

A. Babin, A. Mahalov and B. Nicolaenko (1996), *Global splitting, integrability and regularity of 3D Euler and Navier-Stokes Equations for uniformly rotating fluids*, Europ. J. Mech. B/Fluids, **15**, No. 5, p. 291-300.

A. Babin, A. Mahalov and B. Nicolaenko (1996b), *Global dissipativity and inertial stability of explicit algorithms for Navier-Stokes Equations*, submitted to J. Asymptotic Analysis.

G.L. Browning, W.R. Holland, H.-O. Kreiss and S.J. Worley (1990), *An accurate hyperbolic system for approximately hydrostatic and incompressible oceanographic flows*, Dyn. Atm. Oceans, **14**, p. 303.

C. Devulder, M. Marion, E. S. Titi (1994), *On the rate of convergence of the nonlinear Galerkin method*, Mathematics of Computation.

A. Eden, C. Foias, B. Nicolaenko and R. Temam (1994), *Exponential Attractors for Dissipative Evolution Equations*, R.A.M. Series Vol. 38, John Wiley and Masson.

A. Eden, C. Foias and B. Nicolaenko (1994), *Exponential attractors of optimal Lyapunov dimension*, J. of Dynamics and Diff. Eq., **6**, No. 2, p. 301-323.

J.R. Holton (1992), *An Introduction to Dynamic Meteorology*, 3rd edition, Academic Press.

A. Mahalov (1993), *The instability of rotating fluid columns subjected to a weak external Coriolis force*, Phys. of Fluids A, 5, p. 891–900.

A. Mahalov and P.S. Marcus (1995), *Long Time Averaged Rotating Shallow Water Equations*, Proc. of the First Asian Computational Fluid Dynamics Conference, Hong Kong, eds. W.H. Hui, Y.-K. Kwok and J.R. Chasnov, vol. 3, p. 1227-1230.

A. Mahalov and Y. Zhou (1996), *Analytical and phenomenological studies of rotating turbulence*, Psysics of Fluids, vol. 8, No. 8, p. 2138-2152.

A. Mahalov, E.S. Titi and S. Leibovich (1990), *Invariant helical subspaces for the Navier-Stokes equations*, Arch. Rat. Mech. Anal., 112, p. 193–222.

J. Pedlosky (1987), *Geophysical Fluid Dynamics*, 2nd edition, Springer-Verlag.

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Theses Directed:

1. Weijie Qian, "Exponential Attractors and Inertial Manifolds for Models of Viscoelasticity", Ph.D. Thesis, May 1995, To appear in J. Dynamics Diff. Equations.

2. Nejib Smaoui, "Bifurcations to Chaos in 2D Navier-Stokes Flows", Ph.D. Thesis, May 1994. To appear in Physica D (with D. Armbruster and B. Nicolaenko).